Discrete Structuren)
Hertentamen II, 2007

The problems are to be solved within 3 hrs.

The use of supporting material (books, notes, calculators) is not allowed.

In each problem you can obtain 10 points, i.e. 100 in total. Your partial result for the first 5 problems may be replaced by your grade in the midterm exam ($\times 5$), provided the grade was ≥ 5.5.

Some useful hints:

- Really read these hints.
- Give precise arguments for all your answers.
- You can write in English or Dutch, but in any case use a **readable font!**
- Counterexamples prove that a statement is not true, but positive examples do not prove general validity.
- If you refer to the hand-out sheet, numbers of implications etc. are sufficient.

---

1. Prove that the following proposition

$$(\neg p \to r) \leftrightarrow ((r \to q) \to (p \vee r))$$

is a tautology. Use the form of an *annotated linear proof* (geannoteerd lineair bewijs).

2. Prove (by cases) that $|x + y| \leq |x| + |y|$ for $x, y \in I\!R$.

3. Prove by (infinite) mathematical induction: $\sum_{i=0}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$ for $n \in I\!N$

4. Give an explicit expression for the sequence s_n, defined by

$$\begin{aligned} s_0 &= 1 \\ s_1 &= 1 \\ s_n &= 2s_{n-1} + 2s_{n-2} \quad \text{for } n \geq 2 \end{aligned}$$

5.
(a) Let $s(n)$ ($n \in I\!N$) be a sequence. Define the meaning of $s(n) = O(n)$ and of $s(n) = \Theta(n)$.
(b) Are the following statemens true or false? (Give precise arguments!)

$$2^{2n} = O(2^n) \qquad 2^{n+1} = \Theta(2^n)$$

6. Let the relation $\sim$ on $\mathbb{N}$ be defined by: $m \sim n$ if and only if $5 \mid (m-n)$ (i.e. 5 divides $m-n$). Show explicitly that $\sim$ satisfies the properties of an equivalence relation. What are the equivalence classes of $\sim$?

7.

(a) Show that the proposition

$$[\exists x\, p(x)] \wedge [\exists x\, q(x)] \rightarrow \exists x\, [p(x) \wedge q(x)]$$

is <u>not</u> a tautology. You can do this by giving examples for $p(x)$ and $q(x)$ for which the proposition is false.

(b) Show that the proposition

$$\exists x\, \forall y\; p(x,y) \rightarrow \forall x\, \exists y\; p(x,y)$$

is <u>not</u> a tautology. Again it is sufficient to show that the proposition is false for a particular $p(x,y)$.

8. Let $\boldsymbol{A}$ be the Boolean matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Calculate $\boldsymbol{A} * \boldsymbol{A}$.

(b) Is the relation corresponding to $\boldsymbol{A}$ transitive? Explain your answer!

(c) Which matrices represent the symmetric closure, the reflective closure, and the transitive closure of the relation corresponding to $\boldsymbol{A}$?

9. After having graduated you have been hired by a manufacturer of computer hardware. Your first task is to specify a scheme for the serial number of a new product. You decide on using alphanumerical characters, i.e. the 26 capital letters and the 10 digits.
Your company does not expect to manufacture more then 1000000000 (i.e. 10^9) of these devices. Out of how many alphanumerical characters should the serial number consist, such that there will be a unique serial number for each manufactured device and the serial number of each device is as short as possible?
You should not give the result as a number; it is sufficient to provide an analytic expression, e.g. exp[12] instead of 162754.7914....

10.

(a) How many edges are there in a complete graph with $n = 11$ vertices?

(b) How many edges are there in a binary rooted tree with $n = 23$ vertices?

(c) How many edges are there in a ternary rooted tree with $n = 23$ vertices?

1.	$\neg\neg p \Longleftrightarrow p$	double negation
2a.	$(p \vee q) \Longleftrightarrow (q \vee p)$	commutative laws
b.	$(p \wedge q) \Longleftrightarrow (q \wedge p)$	
c.	$(p \leftrightarrow q) \Longleftrightarrow (q \leftrightarrow p)$	
3a.	$[(p \vee q) \vee r] \Longleftrightarrow [p \vee (q \vee r)]$	associative laws
b.	$[(p \wedge q) \wedge r] \Longleftrightarrow [p \wedge (q \wedge r)]$	
4a.	$[p \vee (q \wedge r)] \Longleftrightarrow [(p \vee q) \wedge (p \vee r)]$	distributive laws
b.	$[p \wedge (q \vee r)] \Longleftrightarrow [(p \wedge q) \vee (p \wedge r)]$	
5a.	$(p \vee p) \Longleftrightarrow p$	idempotent laws
b.	$(p \wedge p) \Longleftrightarrow p$	
6a.	$(p \vee 0) \Longleftrightarrow p$	identity laws[1]
b.	$(p \vee 1) \Longleftrightarrow 1$	
c.	$(p \wedge 0) \Longleftrightarrow 0$	
d.	$(p \wedge 1) \Longleftrightarrow p$	
7a.	$(p \vee \neg p) \Longleftrightarrow 1$	
b.	$(p \wedge \neg p) \Longleftrightarrow 0$	
8a.	$\neg(p \vee q) \Longleftrightarrow (\neg p \wedge \neg q)$	DeMorgan laws
b.	$\neg(p \wedge q) \Longleftrightarrow (\neg p \vee \neg q)$	
c.	$(p \vee q) \Longleftrightarrow \neg(\neg p \wedge \neg q)$	
d.	$(p \wedge q) \Longleftrightarrow \neg(\neg p \vee \neg q)$	
9.	$(p \rightarrow q) \Longleftrightarrow (\neg q \rightarrow \neg p)$	contrapositive
10a.	$(p \rightarrow q) \Longleftrightarrow (\neg p \vee q)$	implication
b.	$(p \rightarrow q) \Longleftrightarrow \neg(p \wedge \neg q)$	
11a.	$(p \vee q) \Longleftrightarrow (\neg p \rightarrow q)$	
b.	$(p \wedge q) \Longleftrightarrow \neg(p \rightarrow \neg q)$	
12a.	$[(p \rightarrow r) \wedge (q \rightarrow r)] \Longleftrightarrow [(p \vee q) \rightarrow r]$	
b.	$[(p \rightarrow q) \wedge (p \rightarrow r)] \Longleftrightarrow [p \rightarrow (q \wedge r)]$	
13.	$(p \leftrightarrow q) \Longleftrightarrow [(p \rightarrow q) \wedge (q \rightarrow p)]$	equivalence
14.	$[(p \wedge q) \rightarrow r] \Longleftrightarrow [p \rightarrow (q \rightarrow r)]$	exportation law
15.	$(p \rightarrow q) \Longleftrightarrow [(p \wedge \neg q) \rightarrow 0]$	reductio ad absurdum

16.	$p \Longrightarrow (p \vee q)$	addition
17.	$(p \wedge q) \Longrightarrow p$	simplification
18.	$(p \rightarrow 0) \Longrightarrow \neg p$	absurdity
19.	$[p \wedge (p \rightarrow q)] \Longrightarrow q$	modus ponens
20.	$[(p \rightarrow q) \wedge \neg q] \Longrightarrow \neg p$	modus tollens
21.	$[(p \vee q) \wedge \neg p] \Longrightarrow q$	disjunctive syllogism
22.	$p \Longrightarrow [q \rightarrow (p \wedge q)]$	
23.	$[(p \leftrightarrow q) \wedge (q \leftrightarrow r)] \Longrightarrow (p \leftrightarrow r)$	transitivity of $\leftrightarrow$
24.	$[(p \rightarrow q) \wedge (q \rightarrow r)] \Longrightarrow (p \rightarrow r)$	transitivity of $\rightarrow$ *or* hypothetical syllogism
25a.	$(p \rightarrow q) \Longrightarrow [(p \vee r) \rightarrow (q \vee r)]$	
b.	$(p \rightarrow q) \Longrightarrow [(p \wedge r) \rightarrow (q \wedge r)]$	
c.	$(p \rightarrow q) \Longrightarrow [(q \rightarrow r) \rightarrow (p \rightarrow r)]$	
26a.	$[(p \rightarrow q) \wedge (r \rightarrow s)] \Longrightarrow [(p \vee r) \rightarrow (q \vee s)]$	constructive dilemmas
b.	$[(p \rightarrow q) \wedge (r \rightarrow s)] \Longrightarrow [(p \wedge r) \rightarrow (q \wedge s)]$	
27a.	$[(p \rightarrow q) \wedge (r \rightarrow s)] \Longrightarrow [(\neg q \vee \neg s) \rightarrow (\neg p \vee \neg r)]$	destructive dilemmas
b.	$[(p \rightarrow q) \wedge (r \rightarrow s)] \Longrightarrow [(\neg q \wedge \neg s) \rightarrow (\neg p \wedge \neg r)]$	

28a.	$\neg 1 \Longleftrightarrow 0$	0-1-wetten
b.	$\neg 0 \Longleftrightarrow 1$	
29a.	$(p \wedge (p \vee q)) \Longleftrightarrow p$	absorptie
b.	$(p \vee (p \wedge q)) \Longleftrightarrow p$	

Equivalenties:

30a.	$p \Longleftrightarrow \forall x\, p$ (x niet vrij in p)	loze kwantificatie
ba.	$p \Longleftrightarrow \exists x\, p$ (x niet vrij in p)	
31a.	$\forall x\, p(x) \Longleftrightarrow \forall y\, p(y)$	herbenoemen van gebonden variabele
b.	$\exists x\, p(x) \Longleftrightarrow \exists y\, p(y)$	
32a.	$\forall x\, \forall y\, p(x,y) \Longleftrightarrow \forall y\, \forall x\, p(x,y)$	kwantorwisseling
b.	$\exists x\, \exists y\, p(x,y) \Longleftrightarrow \exists y\, \exists x\, p(x,y)$	
33a.	$\forall x(p(x) \wedge q(x)) \Longleftrightarrow \forall x\, p(x) \wedge \forall x\, q(x)$	$\forall(\wedge) \Longleftrightarrow \forall \wedge \forall$ ($\forall$ distribueert over $\wedge$)
b.	$\exists x(p(x) \vee q(x)) \Longleftrightarrow \exists x\, p(x) \vee \exists x\, q(x)$	$\exists(\vee) \Longleftrightarrow \exists \vee \exists$ ($\exists$ distribueert over $\vee$)
34a.	$\forall x(p \vee q(x)) \Longleftrightarrow p \vee \forall x\, q(x)$ (x niet vrij in p)	
b.	$\exists x(p \wedge q(x)) \Longleftrightarrow p \wedge \exists x\, q(x)$ (x niet vrij in p)	
35a.	$\neg \forall x\, p(x) \Longleftrightarrow \exists x\, \neg p(x)$	$\neg\forall \Longleftrightarrow \exists\neg$ (De Morgan)
b.	$\neg \exists x\, p(x) \Longleftrightarrow \forall x\, \neg p(x)$	$\neg\exists \Longleftrightarrow \forall\neg$ (De Morgan)

Implicaties:

36.	$\forall x\, p(x) \Rightarrow \exists x\, p(x)$	$\forall \Rightarrow \exists$
37.	$\exists x\, \forall y\, p(x,y) \Rightarrow \forall y\, \exists x\, p(x,y)$	$\exists\forall \Rightarrow \forall\exists$
38a.	$\forall x\, p(x) \vee \forall x\, q(x) \Rightarrow \forall x(p(x) \vee q(x))$	$\forall \vee \forall \Rightarrow \forall(\vee)$
b.	$\exists x(p(x) \wedge q(x)) \Rightarrow \exists x\, p(x) \wedge \exists x\, q(x)$	$\exists(\wedge) \Rightarrow \exists \wedge \exists$